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APPLIED MATHEMATICS AND TURBULENCE MODELLING

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SUMMARY

Turbulence modelling is done traditionally in fluid mechanics departments. However, mathematical tools such as frame invariance, multiple scale expansions and the like are of great help.

We shall demonstrate these facts by applying mathematical and numerical tools to the k–ε model. We shall investigate wall laws, Reynolds hypothesis, positivity of k and ε and flows with multiple scales.

We shall also take this opportunity to review some mathematical results relevant to turbulence modelling.

KEY WORDS: turbulence; modelling; numerical simulations; wall laws; unstructured mesh

1. TURBULENCE

The Navier–Stokes equations for incompressible fluids are capable of turbulence, at least in 3-D:

$$\partial_t u + u\nabla u + \nabla p - \nu\Delta u = f, \quad \nabla \cdot u = 0$$

in a domain Ω, with velocity u given on its boundary Γ and at time zero. In general one is interested in the solution for large times. In practice the flow does not seem to depend much upon initial conditions: the flow around a car, for instance, does not really depend on its acceleration history.

There are many ways 'to forget' initial conditions for a flow. One classifies four regimes.

(a) *Steady* flows.
(b) Time-*periodic* flows.
(c) *Quasi-periodic* flows: the Fourier transform $t \rightarrow F(t) = |u(x, t)|$ has a discrete spectrum and two frequencies at least are not commensurable.
(d) *Chaotic* flows with strange attractors: $\{F(nk)\}_n$ has dense regions of points filling a complete zone of space.

Chaotic flow could be a *mathematical definition of turbulence*. The following points are under study:

(1) Whether there exists attractors, and if so, can we characterize any of their properties? (Hausdorff dimension, inertial manifold, . . . , cf. References 1–4 and the bibliography therein).
(2) Does $u(x, t)$ behave in a stochastic way and if so, by which law? Can we deduce some equations for average quantities such as $\bar{u}$, $\overline{|u|^2}$, $\overline{|\nabla \times u|^2}$, this is the problem of turbulence modelling.

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Here are the main results related to attractors for the Navier–Stokes equations.

Consider the incompressible Navier–Stokes equations with $u_\Gamma = 0$, f independent of t, and Ω a subset of R^2. This system has an attractor whose Hausdorff dimension is between $cRe^{4/3}$ and CRe^2 where $Re = \sqrt{f}\,\mathrm{diam}(\Omega)/\nu$ (cf. References 4–6). These results are interesting because they give an upper bound on the number of points needed to calculate such flows (this number is therefore at least proportional to $\nu^{-9/4}$).

In three dimensions, it is not known that the incompressible Navier–Stokes equations with the same boundary conditions has an attractor but if an attractor exists and is (roughly) bounded by M in $W^{1,\infty}$, then its dimension is less than $CM^{3/4}\nu^{-9/4}$ (cf. Reference 4).

Even if these results are refined the problem of turbulence modelling remains because these upper bounds for the number of computational points (or degrees of freedom are too large for practical applications.

2. WALL LAWS

Turbulence is not the only numerical difficulty for the Navier–Stokes equations. In the vicinity of static walls, the velocity passes from 0 to $O(1)$ over a distance $\delta = O(\sqrt{\nu})$. Numerical simulations will have to take this fact into account by refining the mesh accordingly in the boundary layers. Wall laws are an attempt to remove this constraint.

2.1. The basic idea

The basic idea is to remove boundary layers from the computational domain (Figure 1). Let $\delta(x)$ be the boundary layer thickness above Γ and let

$$B_\delta = \{x - n(x)\lambda\colon x \in \Gamma,\ \lambda \in]0, \delta(x)[\}$$

where n is the outer normal. The computational domain is now $\Omega_\delta = \Omega - B_\delta$ and the new boundary $\Sigma = \partial\Omega_\delta$ replaces Γ.

Of course we need a boundary condition for u on Σ. One possibility is to use a Taylor expansion of u at $x' + \delta(x')n(x')$ which is a point of Γ when $x' \in \Sigma$:

$$u(x' + \delta(x')n(x')) = u(x') + \delta(x')\frac{\partial u}{\partial n}(x') + o(\delta)$$

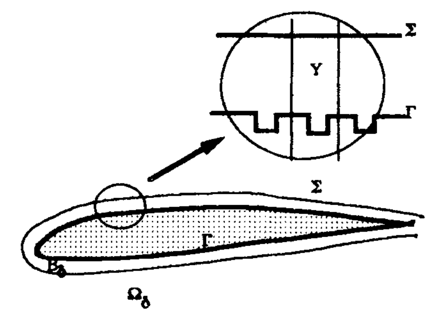

Figure 1. An aerofoil with rough surface. The boundary condition is applied on Σ rather than on Γ. To find the new boundary condition, an auxiliary problem is solved on a representative cell Y with periodic boundary conditions on the vertical sides

Therefore, $u|_\Gamma = 0$ implies

$$u + \delta \frac{\partial u}{\partial n} \approx 0 \quad \text{on } \Sigma.$$

2.2. Wall laws for rough boundaries

However, this works only if $\partial_{nn} u$ is smooth in B_δ. Boundary layers do not satisfy this property everywhere; so another technique must be used namely a multiple scale expansion (homogenization). The method is easy to understand when it is used to account for wall roughness.

So consider an aerodynamic wing profile Γ with very fine periodic irregularities on the surface (Figure 1). Consider a flat surface Σ just above Γ. One seeks a boundary condition on Σ which has the same effect. Following Carrau and Le Tallec,[7] we assume that the flow is stationary and we consider two regions again, B_δ below Σ, i.e. between Γ and Σ, and Ω_δ above Σ. Because B_δ is thin it is conceivable that the flow is somewhat x_1-periodic below Σ. Let Y be a cell of periodicity. If $U_\Sigma \equiv u_{1/\Sigma}$ was known the flow below Σ would be solution of Navier–Stokes equations,

$$u\nabla u + \nabla p - \nu \Delta u = 0, \quad \nabla \cdot u = 0 \quad \text{in } Y$$

$$u|_\Gamma = 0, \quad u_1|_\Sigma = U_{1\Sigma}, \quad u_2|_\Sigma = 0, \quad u, p, x_1 - \text{periodic}$$

The condition $u_2|_\Sigma = 0$ means that Σ is a stream line, which is true as a first approximation.

The solution of this problem depends non-linearly on U_Σ. So $g(U_\Sigma) \equiv \partial u_1/\partial n|_\Sigma$ is some non-linear function of U_Σ. The function $u \to g(u)$ can be tabulated by solving the problem in Y several times with different values of U_Σ.

Above Σ, u is also solution of the Navier–Stokes equations so matching u and ∇u on Σ requires the following boundary conditions:

$$\frac{\partial u_1}{\partial n} = g(u_1), \quad u_2 = 0 \quad \text{on } \Sigma$$

This is called wall law. In effect, it removes the regions of strong gradients from the computational domain at the expense of a more complicated boundary condition. This method can be justified by homogenization in the case of Stokes equations.[8]

2.3. Wall laws for turbulent boundary layers

The previous analysis applies to stationary flows. Consider now the case of an aerofoil. Near the stagnation point a boundary layer develops. In two dimensions it is correctly described by the Falkner–Skan equation. Then the solution becomes unstable and there is a transition to a time-dependent state. Further down stream the solution becomes fully 'turbulent', its numerical simulation becomes almost impossible and turbulence modelling is necessary.

Experiments show some universality in the behaviour of turbulent-attached boundary layers.

More precisely if x_1 denotes the direction of the mean flow parallel to the wall, and x_2 is the co-ordinate in the normal direction, define

$$u^* = \sqrt{\nu \left.\frac{\partial U_1}{\partial x_2}\right|_{x_2=0}}, \quad y^* = \frac{\nu}{u^*} \quad y^+ = \frac{y}{y^*}, \quad u^+ = \frac{U}{u^*}$$

where U is the time averaged velocity. Then experiments show that in the so-called *logarithmic* *layer*, when $20 \leqslant y^+ \leqslant 100$, the scaled mean flow u^+, is a log function of y^+:

$$u^+ = \frac{1}{\chi}\log y^+ + 5{\cdot}5, \quad \chi = 0{\cdot}41.$$

The constant χ is the von Karman constant. This is a non-linear relation between u and $\partial u/\partial n$; it can be used to establish wall laws in place of the auxiliary Navier–Stokes equations of Section 2.2 with periodic conditions in Y. Usually one seeks a δ so that Σ is in the logarithmic layer, and then solves the Navier–Stokes equations with the following boundary conditions, called wall-law:

$$u\cdot n = 0, \quad u\cdot s - \frac{1}{\chi}\sqrt{\nu\left|\frac{\partial u\cdot s}{\partial n}\right|}\left[\log\left(\delta\sqrt{\frac{1}{\nu}\left|\frac{\partial u\cdot s}{\partial n}\right|}\right) + \beta\right] = 0$$

where n is the normal and s a tangent to Σ, $\chi = 0{\cdot}41$, $\beta = 2{\cdot}26$.

Parès[9] showed that the Navier–Stokes equations are well posed with these boundary conditions. In practice, it is necessary to verify, *a posteriori*, that

$$20\sqrt{\nu} \leqslant \delta\sqrt{\left|\frac{\partial u\cdot s}{\partial n}\right|} \leqslant 100\sqrt{\nu}$$

3. REYNOLDS STRESS AND FRAME INVARIANCE

If one writes $u = U + u'$ when U is the mean flow (defined via a filter $\langle\cdot\rangle$) and u' the turbulent fluctuations then the filtered Navier–Stokes equations are

$$\partial_t U + U\nabla U + \nabla P - \nu\Delta U + \nabla\cdot R = 0, \quad \nabla\cdot U = 0$$

The '*Reynolds Stress tensor*' is $R = -\langle u'\otimes u'\rangle$. Let M be a rotation matrix, that is, which verifies

$$(M^{\mathrm{T}}M)_{kj} = M_{ik}M_{ij} = (MM^{\mathrm{T}})_{kj} = M_{ki}M_{ji} = \delta_{kj}$$

Let $x = My$ then, if v denotes the velocity in the y variable,

$$u = \frac{\mathrm{d}x}{\mathrm{d}t} = M\frac{\mathrm{d}y}{\mathrm{d}t} = Mv$$

It is not hard to show that

$$\partial_t u + u\nabla u + \nabla p - \nu\Delta u = M(\partial_t v + v\nabla_y v + \nabla_y p - \nu\Delta_y v)$$

So the Navier–Stokes equations are rotation invariant.

Now it follows from the formulae above that

$$\nabla_x u + \nabla_x u^{\mathrm{T}} = M(\nabla_y v + \nabla_y v^{\mathrm{T}})M^{\mathrm{T}}$$

This allows us to evaluate the Reynolds stress in both frames of reference:

$$\nabla_x\cdot R(\nabla_x u + \nabla_x u^{\mathrm{T}}) = M\nabla_y\cdot[M^{\mathrm{T}}R(M[\nabla_y v + \nabla_y v^{\mathrm{T}}]M^{\mathrm{T}})M]$$

Let u be a solution of Reynolds' equations. Then v will satisfy the same equations in the y-frame if and only if

$$M^{\mathrm{T}}R(M[\nabla_y v + \nabla_y v^{\mathrm{T}}]M^{\mathrm{T}})M = R(\nabla_y v + \nabla_y v^{\mathrm{T}})$$

for all v with $\nabla_y\cdot v = 0$ and all M with $M^{-1} = M^{\mathrm{T}}$. Since all matrices A with zero trace are spanned by $\nabla_y v + \nabla_y v^{\mathrm{T}}$ we must ask that

$$M^{\mathrm{T}} R(MAM^{\mathrm{T}}) M = R(A) \quad \forall A, M \quad \text{with} \quad M^{-1} = M^{\mathrm{T}}, \quad \operatorname{tr} A = 0$$

Proposition 1 (Proof in Reference 10). *To be frame invariant, the only possible form for a symmetric matrix R, function of another symmetric matrix* $A \in R^{d+d}$, is

$$R(A) = a_0 I + a_1 A + \cdots + a_{d-1} A^{d-1}$$

where the a_i *are functions of the invariants of A only.*

Corollary 2. In two dimensions, frame invariance and the assumption that Reynolds' tensor be a function of $\nabla u + \nabla u^{\mathrm{T}}$ *only, imply that Reynolds' equations are*

$$\partial_t u + u\nabla u + \nabla p - \nabla \cdot [\nu_{\mathrm{T}}(|\nabla u + \nabla u^{\mathrm{T}}|)(\nabla u + \nabla u^{\mathrm{T}})] = 0, \quad \nabla \cdot u = 0$$

Remark that in three dimensions the same imply

$$R(\nabla u + \nabla u^{\mathrm{T}}) = aI + \nu_{\mathrm{T}}(\nabla u + \nabla u^{\mathrm{T}}) + \lambda(\nabla u + \nabla u^{\mathrm{T}})^2$$

where a, ν_{T} and λ are functions of $|\nabla u + \nabla u^{\mathrm{T}}|$ and $|(\nabla u + \nabla u^{\mathrm{T}})^2|$ only. This justifies Smagorinsky's turbulence model[11] where ν_{T} is a linear function of its argument. It also says that Reynold's hypothesis is better for 2-D mean flows than for 3-D mean flows.

4. POSITIVITY IN THE k–ε MODEL

Let

$$D_t = \frac{\partial}{\partial t} + u\nabla, \quad E = \frac{1}{2}|\nabla u + \nabla u^{\mathrm{T}}|^2$$

The equations for k, the turbulent kinetic energy and ε, the rate of turbulent energy dissipation, as proposed in Reference 12 are:

$$D_t k - c_\mu \frac{k^2}{\varepsilon} E - \nabla \cdot \left(c_\mu \frac{k^2}{\varepsilon} \nabla k \right) + \varepsilon = 0$$

$$D_t \varepsilon - c_1 k E - \nabla \cdot \left(c_\varepsilon \frac{k^2}{\varepsilon} \nabla \varepsilon \right) + c_2 \frac{\varepsilon^2}{k} = 0$$

with $c_\mu = 0{\cdot}09$, $c_\varepsilon = 0{\cdot}07$, $c_1 = 0{\cdot}126$ and $c_2 = 1{\cdot}92$. They are coupled with the Reynolds equations with $\nu_{\mathrm{T}} = c_\mu k^2/\varepsilon$.

It is possible to justify partially this model. One must assume that turbulence is isotropic at the small scale level, so that means can be replaced by volume averages and that transport by turbulence yields diffusion. The second assumption is that frame invariance applies to R and to $S = \langle \nabla \times u \otimes \nabla \times u \rangle$ but the coefficients may depend also upon k and ε. The third assumption is that $\langle |\nabla(\nabla \times u)|^2 \rangle$ is proportional to ε^2/k. For physical and mathematical reasons, it is essential that the system of partial differential equations for u, p, k, ε yields positive values for k and ε.

4.1. Alternative forms

Now let $\theta = k/\varepsilon$. Then

$$D_t \theta = \frac{1}{\varepsilon} D_t k - \frac{k}{\varepsilon^2} D_t \varepsilon = \theta^2 E(c_\mu - c_1) - 1 + c_2 + \frac{c_\mu}{\varepsilon} \nabla \cdot \left(\frac{k^2}{\varepsilon} \nabla k \right) - c_\varepsilon \frac{k}{\varepsilon^2} \nabla \cdot \left(\frac{k^2}{\varepsilon} \nabla \varepsilon \right)$$

Thus

$$D_t\theta - \theta^2 E(c_\mu - c_1) + 1 - c_2 = (c_\mu - c_\varepsilon)\theta^2\Delta k + c_\varepsilon k\theta\Delta\theta$$
$$+ 4\,\mathrm{sign}(k)(c_\mu - c_\varepsilon)\theta^2|\nabla\sqrt{|k|}|^2 + (c_\mu + 2c_\varepsilon)\theta\nabla k\cdot\nabla\theta - c_\varepsilon k|\nabla\theta|^2$$

4.2. Positivity and exponential growth without viscosity

If there were no viscous terms in the equations for k and ε then the θ equation would be an autonomous ordinary differential equation on the steam lines:

$$D_t\theta - \theta^2 E(c_\mu - c_1) + 1 - c_2 = 0$$

It has always a positive bounded solution when the initial and boundary data are positive because $c_\mu < c_1$ and $c_2 > 1$. Similarly, in the absence of viscous terms the equation for k reduces to

$$D_t \log k = c_\mu\theta E - \frac{1}{\theta}$$

which has always a positive solution for positive data but it grows exponentially when $c_\mu\theta^2 E > 1$.

4.3. Positivity in the case of Dirichlet boundary data

Let us analyse the system for θ, k by using the maximum principle. Assume positive initial data and positive Dirichlet boundary data and suppose that the solution is continuously differentiable. Let t^* be the first instant for which θ reaches zero and assume that k is positive on $[0, t^*]$. Let x^* be the point where this happens. Because we have assumed θ and k smooth and because x^* cannot be on the boundary (where θ is given), x^* must be a minimum for θ so we have

$$\nabla\theta(x^*, t^*) = 0, \quad \theta(x^*, t^*) = 0 \quad (\text{and if } \theta\in C^2\text{: } \Delta\theta(x^*, t^*) \geqslant 0)$$

By writing the θ equation at this point, we obtain:

$$\partial_t\theta = c_2 - 1 > 0$$

This is a contradiction; indeed $t \to \theta(x^*, t)$ has been decreasing up to t^* so $\partial_t\theta$ *was* negative and suddenly it becomes positive; thus θ is not continuously differentiable (in any case it grows again away from zero).

Now let $x^0(t)$ be the minimum of $k(x, t)$ in x. If $x^0(t)$ is on the boundary then the minimum being positive, k is positive at time t. If $x^0(t)$ is not on the boundary then ∇k is zero and $\Delta k \geqslant 0$ at $\{x^0(t), t\}$. So the k-equation:

$$\partial_t k + U\nabla k - c_\mu k\theta E - c_\mu k\theta\Delta k - c_\mu\nabla(\theta k)\cdot\nabla k + \frac{k}{\theta} = 0$$

yields

$$\partial_t k \geqslant k\left(c_\mu\theta E - \frac{1}{\theta}\right) \quad \text{at } \{x^0(t), t\}.$$

Now let $\kappa(t) = k(x^0(t), t)$. By construction $\partial_t\kappa = \partial_t k$ so the equation above implies

$$\kappa(t) = \min_x k(x, t) \geqslant \kappa(0)\exp\left(\int_0^t \left[c_\mu\theta E - \frac{1}{\theta}\right](x^0(t'), t')\,dt'\right)$$

Therefore k is strictly positive.

The previous analysis has one defect: it assumes that the solution exists and is smooth. It seems hard to prove that it is so. However Lewandowski *et al.*[13,14] have established a partial result for a simpler model. Their analysis is based on a new variable $\varphi = \varepsilon^2/k^3$ which satisfies the following:

$$D_t\varphi + (3c_\mu - 2c_1)E\varphi\theta + (2c_2 - 3)\frac{\varphi}{\theta}$$

$$= 2\frac{\varepsilon}{k^3}\nabla\cdot\left(c_\varepsilon\frac{k^2}{\varepsilon}\nabla\varepsilon\right) - 3\frac{\varepsilon^2}{k^4}\nabla\cdot\left(c_\mu\frac{k^2}{\varepsilon}\nabla k\right)$$

$$= (3c_\mu - 2c_\varepsilon)\frac{\Delta\varphi}{\varphi\theta} + 6(c^\mu - c_\varepsilon)\frac{\Delta\theta}{\theta^2} + (21c_\mu - 20c_\varepsilon)\frac{\nabla\theta\cdot\nabla\varphi}{\theta^2\varphi}$$

$$- (9c_\mu - 6c_\varepsilon)\frac{|\nabla\varphi|^2}{\varphi^2\theta} - (24c_\mu - 30c_\varepsilon)\frac{|\nabla\theta|^2}{\theta^3}$$

The advantage of this equation over the k-equation is that in the absence of viscous terms (the right-hand side), it is explicit in $\log\varphi$:

$$D_t\log\varphi = -(3c_\mu - 2c_1)E\theta - (2c_2 - 3)\frac{1}{\theta}.$$

Hence φ is always decreasing because $3c_\mu - 2c_1 = 0{\cdot}0188$ and $2c_2 - 3 = 0{\cdot}84$.

5. A NUMERICAL METHOD FOR k–ε

By using what we know about the positivity of the k–ε model, it is possible to build a stable multistep scheme which involves only linear systems at each time step and is yet unconditionally stable.

At every time step, the Navier–Stokes equations are solved with ν_T and the boundary conditions computed at the previous time step. The equations for k–ε are solved by a multistep algorithm involving one step of convection and one step of diffusion. However in this case the convection step is performed on k, θ or φ, θ rather than on k, ε.

The equation for θ is integrated without diffusion:

$$(\theta_h^{m+1/2}, w_h) + (\theta_h^m\theta_h^{m+1/2}E_h^m, w_h)[c_1 - c_\mu]\delta t = (\theta_h^m o(X_h^m, w_h) + (c_2 - 1, w_h)\delta t$$

with $\theta_h^m = k_h^m/\varepsilon_h^m$. The equation for $\varphi = \varepsilon^2/k^3$ (see Section 3) is also integrated without diffusion:

$$(\varphi_h^{m+1/2}, w_h) + \delta t((3c_\mu - 2c_1)E_h^m\varphi_h^{m+1/2}\theta_h^m, w_h) + \delta t\left((2c_2 - 3)\frac{\varphi^{m+1/2}}{\theta_h^{m+1/2}}, w_h\right)$$

$$= (\varphi_h^m oX_h^m, w_h), \quad \forall w_h \in W_h$$

Then $k_h^{m+1/2}$, $\varepsilon_h^{m+1/2}$ are computed from the formulae

$$k_h^{m+1/2} = \frac{1}{\varphi_h^{m+1/2}(\theta_h^{m+1/2})^2}, \quad \varepsilon_h^{m+1/2} = \frac{k_h^{m+1/2}}{\theta_h^{m+1/2}}$$

Finally the diffusion step is applied to k and ε,

$$(k_h^{m+1}, w_h) + \delta t c_\mu \left(\frac{k_h^{m^2}}{\varepsilon_h^m} \nabla k_h^{m+1}, \nabla w_h \right) = (k_h^{m+1/2}, w_h)$$

$$(\varepsilon_h^{m+1}, w_h) + \delta t c_\varepsilon \left(\frac{k_h^{m^2}}{\varepsilon_h^m} \nabla \varepsilon_h^{m+1}, \nabla w_h \right) = (\varepsilon_h^{m+1/2}, w_h)$$

for all $w_h \in Q_{oh}$; $\varepsilon_h^{m+1} - \varepsilon_{\Gamma h} \in W_{oh}$, $k_h^{m+1} - k_{\Gamma h} \in W_{oh}$.

Proposition 3. With Lagrangian finite element of degree 1 on a triangulation without obtuse angle and with mass lumping on the first and last integrals in the diffusion step, the above scheme cannot produce negative values for k_h^{m+1} *and* ε_h^{m+1}.

Proof. Each step produces positive values only. It is known[15] that the maximum principle holds in the discrete case, with P^1 finite elements and triangles with sharp angles, for coercive operator, like the one in the diffusion step. □

Remark. This analysis can be extended to compressible k–ε.[16]

Numerical results are shown at the end of the paper with this method for the transonic bump and for a NACA0012 at high degree of incidence.

6. TURBULENT TRANSPORT

Reynolds' hypothesis is the key to turbulence modelling. Whether transport by a turbulent velocity field yields diffusion is being heavily studied by applied mathematicians.[17]

6.1 The linear problem

The same problem arise in passive turbulent transport. Consider the linear convection diffusion equation:

$$\partial_t c + (U + u')\nabla c - \mu \Delta c = 0$$

It is known[18] that if u' is a random stationary mixing process with $U \gg \sqrt{\langle |u'|^2 \rangle}$ then $C = \langle c \rangle$ satisfies

$$\partial_t C + (U + u')\nabla C - \nabla \cdot ((\mu + M)\nabla C) = 0$$

with $M = \frac{1}{2}\int_{-\infty}^{\infty} \langle u'(x + Ut) \otimes u'(x) \rangle \, dt$. It is conjectured that the result holds also when U and u' are of the same order; however the formula for M is more complicated. There are counter examples to this results when u is not mixing or has zero mean.[19,20] Extensions to non-linear cases such as Navier–Stokes equations is an open problem, in general (see References 21–24).

When oscillations are periodic or quasic-periodic and bounded in L^2, the problem of finding an equation for the mean of the solution of a convection equation can be solved with tools of functional analysis such as compensated compactness[25,26] Young measures[27,28] and H-measures;[29] but the results can be derived formally also by multiple-scales asymptotic expansions.

6.2. Compressible Euler in 1-D

In one dimension Serre[30] solved the closure problem for the compressible Euler equations completely. Euler equations can be written in terms of entropy variables, i.e.

$$(\partial_t + u\partial_x)u + \frac{1}{\rho}\partial_x p = 0, \quad (\partial_t + u\partial_x)p + \rho c^2 \partial_x u = 0, \quad (\partial_t + u\partial_x)S = 0$$

where $c^2 = \gamma(\gamma - 1)e$ and $S = \log(p\rho^{-\gamma})$. The system is integrated on $R \times]0, T'[$ with initial conditions

$$\rho(0, x) = \rho^0(x, y), \quad u(0, x) = u^0(x, y), \quad e(0, x) = e^0(x, y)$$

where y is a parameter. The idea is to solve the system for all y instead of just $y = x/\varepsilon$. Then the solution depends upon y and the problem is to find a system of equations for $\bar{u}, \bar{p}, \bar{S}$, where

$$\bar{u} = \lim_{Y\to\infty} \frac{1}{2Y}\int_{-Y}^{Y} u(x, y)\,dy$$

Under certain conditions (no shocks, integrability. . .) the answer is

$$(\partial_t + \bar{u}\partial_x)\bar{u} + \frac{1}{\langle\rho\rangle}\partial_x \bar{p} = 0$$

$$(\partial_t + \bar{u}\partial_x)\bar{p} + \frac{1}{\langle\rho^{-1}c^{-2}\rangle}\partial x\bar{u} = 0$$

$$\partial_t\langle\rho\varphi(S)\rangle + \partial_x(\bar{u}\langle\rho\varphi(S)\rangle) = 0 \quad \forall\varphi.$$

The filter $\langle\cdot\rangle$ is defined on functions of p, S by

$$\langle f\rangle = \lim_{Y\to\infty}\frac{1}{2Y}\int_{-Y}^{Y} f(\bar{p}, S(y))\,dy$$

Hence, with $\rho = p^{1/\gamma}e^{S/\gamma}$

$$\langle\rho\rangle = \lim_{Y\to\infty}\frac{1}{2Y}\int_{-Y}^{Y} \bar{p}^{1/\gamma}\, e^{S(y)/\gamma}\,dy$$

and similarly for $\rho^{-1}c^{-2}$ and $\rho\varphi(S)$, by the formulae:

$$\rho\varphi(S) = \varphi(S)p^{1/\gamma}e^{S/\gamma}, \quad \frac{1}{\rho c^2} = \frac{1}{\gamma p}$$

It is an integro-differential system which must be analysed further for practical applications. In particular when the intial conditions are quasi-periodic in y, the quantities $\langle\rho\rangle$ must be evaluated. Extension of the method to several space dimensions and to the Navier–Stokes equations can be found in References 30, 31 and 32.

6.3. Multiple scales asymptotics

Finally multi-scale asymptotics can be helpful to analyse turbulent transport by the Navier–Stokes equations of initial 'turbulent' conditions. Consider

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - \varepsilon^2 \mu \Delta u^\varepsilon + \nabla p^\varepsilon = 0$$

$$\nabla \cdot u^\varepsilon = 0 \quad \text{in } R^3 \times]0, T[$$

$$u(x, 0) = u_0(x) + \varepsilon^{1/3} w_0\left(x, \frac{x}{\varepsilon}\right) \quad \text{in } R^3$$

where $w_0(x, y)$ is almost periodic in y and has zero y-mean.

Here ε is a length scale and not the rate of turbulent energy dissipation which will be called e; the turbulent kinetic energy k will also be called q and h will denote helicity. The choice $\mu\varepsilon^2$ for the vanishing viscosity is based on Kolmogorov's scales. At time zero the kinetic turbulent energy is $\varepsilon^{2/3} q_0 = \varepsilon^{1/3} \langle |w_0|^2 \rangle$ and the rate of turbulent dissipation is $\varepsilon^{2/3} e_0 = \mu \varepsilon^{2/3} \langle |\nabla_y \times w_0|^2 \rangle$. The helicity is $\varepsilon^{-1/3} h_0 = \varepsilon^{-1/3} \langle w_0 \cdot \nabla_y \times w_0 \rangle$.

If w_0 is isotropic and h_0/q_0 is constant then it can be shown[33,34] that u is approximatively the solution of

$$\partial_t u + (u \cdot \nabla) u + \nabla p + \varepsilon^{2/3} \nabla \cdot [q^0(a(x, t)) m(i) \nabla a \nabla a^{\mathrm{T}}] = 0, \quad \nabla \cdot u = 0$$

$$\partial_t a + u \cdot \nabla a = 0, \quad a(x, 0) = x$$

with $i = \sum_{l,j} \partial_j a_l \partial_j a_l$, $\gamma_0 \approx \frac{1}{9}$, $\beta_0 \approx \frac{1}{3}$ and

$$m(i) = \frac{\beta_0}{(1+i)^2} \mathrm{e}^{-\mu \gamma_0 t} \mathrm{e}^{(2\beta_0/3)(i - 2/1 + i)}$$

This result is established without closure hypothesis. It is a two equations model like k–ε but it involves the Lagrangian length scale a which is the position at time t of a particle which was at x at time 0.

Although we do not reproduce the proof here, it agrees with the k–ε model for an homogeneous free decaying turbulence, thus validating the ε^2/k hypothesis mentioned earlier. On the other hand, it seems to disprove Reynolds hypothesis because Reynolds' tensor is a function of $\nabla a^{\mathrm{T}} \nabla a$ instead of $\nabla u + \nabla u^{\mathrm{T}}$. If the asymptotic expansion is pursued further then at the next level a viscous term, function of ∇u is found. Nevertheless, it indicates the Reynolds hypothesis may not be the first order effect in a transient flow.

REFERENCES

1. D. Ruelle and F. Takens, 'On the nature of turbulence', *Comm. Math. Phys.*, **20,** 167–192 (1971).
2. J. M. Ghidaglia, 'On the fractal dimension of attractors for viscous incompressible flows', *SIAM J. Numer. Anal.*, **17,** 1139–1157 (1986).
3. P. Constantin, C. Foias, O. Manley and R. Temam, 'Determining modes and fractal dimension of turbulent flows', *J. Fluid Mech.*, **150,** 427–440 (1985).
4. P. Bergé, Y. Romeau and Ch. Vidal, *De l'ordre dans le chaos, Hermann, Paris,* 1984.
5. D. Ruelle, 'Les attracteurs étranges'. *La recherche*, **108,** 132 (1980).
6. D. Ruelle, 'Characteristic exponents for a viscous fluid subjected to time dependent forces', *Comm. Math. Phys.*, **93,** 285–300 (1984).
7. A. Carrau and P. Le Tallec, 'Modelisation numerique d'un ecoulement sur paroi-rugeuse', *Thèse de Doctorat,* University of Bordeaux, 1992.
8. C. Conca, 'Etude d'un fluide traversant une paroi preforée', *J. Math Pures Appl.*, **66,** 45–69 (1987).
9. C. Parès, 'Un traitement faible par élément finis de la condition de glissement sur une paroi pour les équations de Navier–Stokes', *Note C.R.A.S.*, **307 I,** 101–106 (1988).
10. Ph. Ciarlet, *Elasticité Tridimensionnelle,* Masson, Paris, 1986.

11. J. Smagorinsky, 'General circulation model of the atmosphere', *Mon. Weather Rev.*, **91,** 99–164 (1963).
12. B. E. Launder and D. B. Spalding, *Mathematical Models of Turbulence*, Academic Press, New York, 1972.
13. R. Lewandowski and B. Mohammadi, 'Existence and positivity results for the ϕ–θ model, and a modified k–ε turbulence model', *Math. Model Methods Appl. Sci.*, **3**, 195–215 (1993).
14. R. 'Lewandowski, Modlèles de turbulences et équations paraboliques,' CRAS, 1993, to appear.
15. Ph. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
16. B. Mohammadi and O. Pironneau, *Analysis of the k–ε Model*, Wiley, New York, 1993.
17. V. Maslov, 'Homogenization of Navier–Stokes equations', *Seminaire College de France*, ed. J. L. Lions, Pitman, 1987.
18. H. Kesten, G. Papanicolaou, 'A limit theorem for stochastic acceleration', *Comm. Math. Phys.*, **78,** 19–63 (1980)
19. S. Childress and A. Soward, 'Scalar transport and alpha effect for a family of cat's eye flows', *J. Fluid Mech.*, **205,** 99–133 (1989).
20. M. Avelaneda and A. Majda, 'Mathematical models with exact renormalization for turbulent transport', *Comm. Math. Phys.* (1992).
21. G. Papanicolaou and O. Pironneau, 'On the asymptotic behaviour of motion in random flow, in Arnold–Lefever (eds.), *Stochastic Nonlinear Systems*, Springer, Berlin, 1981.
22. M. Isichenko, 'Percolation, statistical topography and transport in random media,' *Rev Mod. Phys.*, **64,** 961 (1992).
23. T. Hou and X. Xin, 'Homogenization of linear transport equations, with oscillatory vector fields', *SIAM J. Appl. Math.*, **52**(1), 34–45 (1992).
24. V. Maslov, 'Homogenization of Navier–Stokes equations', in H. Brezis and J. L. Lione (eds.), *Seminaire du Collège de France*, Pitmann, London, 1987.
25. L. Tartar, 'Compensated compactness and applictions', *Proc. Nonlinear Analysis and Mechanics symposium*, Pittman RNM 39, 1979.
26. F. Murat, 'Compacité par compensation,' *Ann. Sc. Norm. Sup. Pisa*, **5,** 489–507 (1978).
27. L. Young, 'Lectures on the calculus of variations and optimal control', W. B. Saunders, Philadelphia, 1969.
28. R. diPerna and A. Majda, 'The validity of nonlinear geometric optics for weak solutions of conservation laws', *Comm. Math. Phys.*, **98,** 313–347 (1985).
29. L. Tartar, 'H-measures and small amplitude homogenization, in R. Kohn (ed.), *Proc. SIAM workshop on Random media and composites*. p. 89–99, SIAM books 1989.
30. L. Tartar, 'Compensated compactness and applications'. *Proc. Nonlinear Analysis and Appl.* Pitman, 1979.
31. D. Serre, 'Oscillations nonlineaire des systeme hyperbolique' *Ann. Inst. H. Poincaré*, Vol. 8, no. 3, pp 351–417, 1991.
32. D. Serre, 'Quelques methodes d'étude de la propagation d.oscillations hyperboliques nonlinéaires', Seminaire no 20 'Equations aux dérivées partielles', Ecole Polytechnique Mars, 1991.
33. T. Chacon, D. Franco and F. Ortegon, 'Helicity in turbulence modeling by homogenization', *Comput. Fluid Dyn.* (1993).
34. D. Mclaughin, G. Papanicolaou and O. Pironneau, 'Convection of microstructures', *SIAM J. Numer. Anal.*, **45,** 780–796 (1982).